A Research Proposal

on

A Study of Generalized Weighted Composition Operators on Weighted Hardy Spaces

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PUNJAB

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INTRODUCTION:

Let X be a non empty set and F be a field. Suppose $V(X)$ is a topological vector space of complex valued functions on X over the field F. Let $\tau : X \to X$ is a mapping such that $f\circ\tau \in V(X)$ whenever $f \in V(X)$, then we can define a composition transformation $C_\tau : V(X) \to V(X)$ by $C_\tau f = f\circ\tau$ for every $f \in V(X)$.

In case C_τ is continuous, we call it a composition operator induced by τ. Further, if $\tau : X \to F$ is a function such that $\tau . f \in V(X)$ whenever $f \in V(X)$, then a multiplication transformation $M_\tau : V(X) \to V(X)$ is defined by the equation $M_\tau f = \tau . f$ for every $f \in V(X)$. In case M_τ is continuous, we shall call it a multiplication operator induced by τ. The class of multiplication operators and the class of composition operators are combined to yield another important class of operators called weighted composition operators. Thus a weighted composition operator is of the type $M_\tau C_\tau$ or $C_\tau M_\tau$ and we usually denote it by $W_{\tau,\rho}$. A transformation of the type $W_{\tau,\rho} : V(X) \to V(X)$ defined by $W_{\tau,\rho} f = \tau . f\circ\rho$. for every $f \in V(X)$ is called weighted composition transformation. If it is continuous, we shall call it a weighted composition operator induced by the pair (τ,ρ).

Multiplication operators play an important role in the study of operators on Hilbert spaces. For example, every normal operator on Hilbert space has a non trivial subspace. This is true because every normal operator is unitarily equivalent to a multiplication operator and every multiplication operator has a non trivial invariant subspace. A well known example of Multiplication operator is that of diagonal operator D on l^2, which is given by $D(x_1, x_2,.....) = (\tau_1 x_1, \tau_2 x_2,.....)$ where (τ_n) is a bounded sequence of complex numbers. Unilateral shifts, left shifts, bilateral shifts and translation on space $L^2(G)$, where G is a topological group

are well known examples of composition operators.

WEIGHTED HARDY SPACES:

Let $\beta = \{\beta_n\}_{n=0}^{\infty}$ be a sequence of strictly positive real numbers with $\beta_0 = 1$. Consider the space of sequence $f = \{f(n)\}_{n=0}^{\infty}$ of complex numbers such that

$$\|f\|^2 = \|f\|_\beta^2 = \sum_{n=0}^{\infty} |f(n)|^2 \beta_n^2 < \infty.$$

We shall use the notation $f(z) = \sum_{n=0}^{\infty} f(n)z^n$, whether or not series converges for any complex value z, these series are called formal power series. We shall denote the space of all formal series $\sum_{n=0}^{\infty} f(n)z^n$ for which $\|f\|^2 < \infty$ by $H^2(\beta)$, the space $H^2(\beta)$ are Hilbert spaces with the inner product $\langle f, g \rangle = \sum_{n=0}^{\infty} f(n)\overline{g(n)}\beta_n^2$.

Let $e_k(z) = z^k$ and $\hat{e}_k(z) = \frac{z^k}{\beta_k} = \frac{e_k(z)}{\beta_k}$, Clearly $\{e_k\}_{k=0}^{\infty}$ is an orthogonal basis for $H^2(\beta)$ and $\hat{e}_k : k \in N_o$ where $N_o = \{0,1,2,3,.....\}$ is an orthonormal basis for $H^2(\beta)$.

The classical Hardy, Bergman and Dirichlet spaces on the unit disk usually defined by integral may be thought of in this way. The usual Hardy space H^2 has norm defined by $\|f\|^2 = \lim_{r \to 1} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^2 \frac{d\theta}{2\pi}$

and it is the case of $H^2(\beta)$ when $\beta(n) = 1$ for all $n \in N_o$.The Bergman space has a norm defined by

$$\|f\|^2 = \int_D |f(x+iy)|^2 \frac{dxdy}{\pi}$$

This is the case of $H^2(\beta)$ when $\beta(n) = \frac{1}{\sqrt{n+1}}$ for all $n \in N_o$.The Dirichlet space has a norm defined by

$$\|f\|^2 = \sum_{k=0}^{\infty} |f(k)|^2 + \int_D |f'(x+iy)|^2 \frac{dxdy}{\pi}$$

and this is the case of $H^2(\beta)$ when $\beta(n) = \sqrt{n+1}$ for all $n \in N_o$.If we take $\beta(n)^2 = n!$, $H^2(\beta)$ consist of entire function and it is called Fischer space. If then

the sequence of ratios $\frac{\beta_n}{\beta_{n-1}}$ decreases to zero as n increases to infinity, then space $H^2(\beta)$ is a Hilbert space of entire functions and every sequence convergent in norm

of $H^2(\beta)$ is uniformly convergent on the compact subset of the plane. The symbol β

$H(\beta)$ denote the set of formal power series $\varphi(z) = \sum_{n=0}^{\infty} \varphi(n) z^n$ such that $\beta_0 = 1$

$H^2(\beta) \subset H^2(\beta)$.We shall denote by Ω, subset of C such that $f \in H^2(\beta)$ is every

analytic on Ω. All weighted Hardy spaces contain spaces of polynomials as a dense linear space. If $\lim_{n\to\infty} \beta_n^{1/n} = 1$ or $\lim_{n\to\infty} \frac{\beta_{n+1}}{\beta_n} = 1$, the space $H^2(\beta)$ contains analytic functions in the unit disk. However none of the spaces contains all analytic functions in the unit disk. Let w be a point on the open unit disk. Define $K_w(z) = \sum_{n=0}^{\infty} \frac{\overline{w}^n}{\beta_n^2} z^n$,

Then the function K_w is a point evaluation for $H^2(\beta)$. Clearly it is easy to see that K_w is in $H^2(\beta)$ and

$\|K_w\|^2 = \sum_{n=0}^{\infty} \frac{|w|^{2n}}{\beta_n^2}$, Thus $\|K_w\|$ is an increasing function of $|w|$. For $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$

$\langle f, K_w \rangle = \sum_{n=0}^{\infty} \frac{a_n w^n}{\beta_n^2} \beta_n^2 = \sum_{n=0}^{\infty} a_n w^n = f$ for all f. Clearly $K_o = 1$(the function identically equal to 1) . By the symbol $K(w,z)$ we shall mean $K_w(z)$.

Let $B(H)$ denote the Banach Algebra of all bounded linear operators from Hilbert space H into itself. An operator $A \in B(H)$ is called

(i) Hermitian if $A = A^*$

(ii) Normal if $A^*A = AA^*A^*$

(iii) Quasinormal if $AA = AA^*A$

(iv) Hyponormal if $AA^* = A^*A$

(v) Isometry if $A^*A = I$

(vi) Unitary if $A^*A = I = AA^*$

(vii) Compact , if $\overline{A(E)}$ is compact for every bounded subset E of H .

(viii) Fredhlom , if Ker A and Ker A^* are finite dimensional and range of A is closed.

Definition: A generalized composition operator $C_{\varphi}^{d} : H^2 \to H^2$ is defined by $C_{\varphi}^{d} f = f \circ \varphi$ where f' is the derivative of f and d is the differential operator.

Definition: A generalized multiplication operator $M_{\psi}^{d} : H^2 \to H^2$ is defined by $M_{\psi}^{d} f = \psi . f'$.

Definition: A generalized weighted composition operator $W_{\psi,\varphi}^{d} : H^2 \to H^2$ is defined by $W_{\psi,\varphi} f = \psi . f' \circ \varphi$. If we take $\psi = z$ for every $z \in D$, Then $W_{\psi,\varphi} = M_{\psi}^{d} C_{\varphi}^{d}$ which is a generalized multiplication operator.

REVIEW OF LITERATURE

Composition operators found its first appearance in 1871 in a paper of Schroeder[17] where it was asked to find a function f and λ number such that $f \circ T = \lambda f$ for every given function T and for every z in the appropriate domain. If z varies in the open unit disk and T is an analytic function, then a solution has been obtained by Koeings[8] in 1884. Afterwards in 1925 these operators found their applications in Littlewood's[12] subordination theory. In early's 1930 the composition operators were used to study problems in mathematical physics and classical mechanics, especially worth mention is the work of Koopman[10] on classical mechanics. In those days these operators were known as the substitution operators or translation operators. Banach[1] himself made use of these operators to study the isometries on Banach spaces of continuous fnctions. In the 1940's composition operators appeared in the work of Neumann and Halmos[14] in the study of ergodic transformations. In 1966 Choksi[2] studied unitary composition operators and continued the study in later years.

The history of systematic study of the composition operators is not that old. It was started in 1968 by Nordgren[15]. In his paper Nordgren studied composition operators on L^2 of the unit circle by inner functions. Perhaps Schwartz[18] wrote his first Ph.D dissertation in 1969 on composition operators on H^p spaces . In 1969, Ridge[16] also wrote his Ph.D dissertation on composition operators on L^p spaces followed by

Singh[26] , who completed his Ph.D dissertation on composition operators in 1972

under the supervision of Prof. Eric A. Nordgen. Then the work on composition operators is than followed by many mathematicians in several directions.

The three main directions are given below

(a) The underlying space X is a measure space and the inducing maps are measurable transformations,

(b) The underlying spaces X is a topological space and the inducing maps are continuous functions,

(c) The underlying spaces X is a region in C or C^n and the inducing maps are holomorphic functions.

Extensive literature is now available on composition operators, three main books on composition operators are Shapiro[19], Singh and Manhas[28], Cowen and MacCluer[4].

The systematic study initiated in the 1970's has been continued and extended in several directions during the last decade. Worth mention are some names such as Kumar[11], Komal[9],Sharma[21],MacCluer[13], etc.,explored the properties of the composition operators on different function spaces. Jamison and Rajagopalan[7] made systematic studies of these operators on several spaces of continuous functions. Gunatillake [6], Shapiro and Taylor[20], Zorboska[34] have done commendable work on spaces of analytic functions.Cowen and MacCluer[5],Yousefi[30],Sharma and Komal[22] introduced the study of composition operators on different spaces. Zhu[32,33] in 2009 had discussed the compactness and boundedness of Generalized weighted composition operators on weighted Bergman spaces, Sharma and Komal[22,23,24] in 2011 and 2012 have done work on generalized composition operator and generalized multiplication operator on weighted hardy space.

OBJECTIVES/SCOPE OF STUDY

During our Research work we proposed to study the following problems on generalized weighted composition operators.

. To characterize the adjoint of generalized weighted composition operators on weighted hardy spaces.

. To characterize the Normality, Boundedness, Compactness of Generalized weighted composition operators on weighted hardy spaces.

. To characterize the condition for a Generalized weighted composition operators to be a Fredhlom on weighted hardy spaces.

. Computation of Spectra of Generalized weighted composition operators on weighted hardy spaces.

. Computation of Numerical Range of Generalized weighted composition operators on weighted hardy spaces.

PROPOSED METHODOLOGY:

The techniques used by Singh and Manhas[29], Singh and Summers[31],Cowen[3],Singh and Komal[27],Shield[25],Zorboska[35] will be used to study the properties of generalized weighted composition operator on the weighted Hardy spaces.

The Banach Steinhaus theorem, open mapping and closed graph theorems will be used to prove the continuity of these operators . We shall make use of Radon Nikodyon theorem to study generalized multiplication operators and composition operators on the L^p spaces of vector valued functions.

The expectation operators will be employed in characterizing bounded compact and Fredhlom generalized weighted composition operators.

The thesis will be divided into three phases

Phase1: This phase includes the study of recent books and literature survey to collect the detail information about the composition operators, multiplication operators and weighted composition operators.

Phase2: The purposed research work will be done.

Phase3: The proposed work will be presented in the form of thesis consisting of four chapters. The first chapter will be introductory in nature. The remaining three chapters will contains the results regarding the generalized weighted composition operator on weighted Hardy spaces.

Paper Presented in National Conferences:

1. Differential and Antidifferential Operators on Weighted Hardy Space in Bhartiya Vigyan Sammelan at Lovely Professional University, Phagwara held on Oct-11 to 14,2012.

Attended National and International Conferences/Seminar:

1. Attended Seminar on Recent Trends in solid Mechanics & Functional Analysis at Indus International University, Una ,on 12th December, 2011.
2. Attended National Conference on Algebra, Analysis and their Application at Govt. M.A.M. College, Jammu, on 18th and 19th Aug. 2012.
3. Attended International Conference Differential Geometry, Functional Analysis and Application at Jamia Millia Islamia University, New-Delhi on Sept-8 to 10.2012.
4. Attended Bhartiya Vigyan Sammelan at Lovely Professional University, Phagwara held on Oct-11 to 14,2012.

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